

Torsors

Ex. For $a \in \mathbb{R}_{>0}$, set $S_a := \{z \in \mathbb{C} \mid |z| = a\}$

S_1 is a gp., it acts simply transitively on each S_a

$\leadsto$ each S_a is an S_1 -torsor.

Def. For a sheaf of groups G ,

- a G -torsor is a sheaf E equipped with a simply transitive right G -action, i.e., s.t. E is locally isomorphic to G equipped with the right translation action ($\Leftrightarrow$ s.t. $E \times G \xrightarrow{(e,g) \mapsto (e, eg)} E \times E$ and locally E has a section);
- a G -torsor E is trivial if it is isomorphic to G equipped with the right translation action ($\Leftrightarrow$ if E has a global section (= a trivialization)).

Principle For a geometric object X ,

$$\{\text{forms of } X\} / \sim \xleftrightarrow{\sim} \{\text{Aut}(X)\text{-torsors}\} / \sim$$

- 1) $\{\text{vector bundles of rank } n\} / \sim \xleftrightarrow{\sim} \{GL_n\text{-torsors}\} / \sim$
 - 2) $\{\text{Serre-Brauer schemes of dim } n\} / \sim \xleftrightarrow{\sim} \{PGL_{n+1}\text{-torsors}\} / \sim$
projective smooth $\begin{smallmatrix} X \\ \downarrow \\ S \end{smallmatrix}$ that are étale locally on S iso. to $\begin{smallmatrix} \mathbb{P}_S^n \\ \downarrow \\ S \end{smallmatrix}$
 - 3) $\{\text{nondegenerate quadratic forms of rank } n\} / \sim \xleftrightarrow{\sim} \{O_n\text{-torsors}\} / \sim$
- The classical case of the Grothendieck-Serre conjecture

Goal of these lectures: explain some aspects of

Main Thm. (Boutin - Č. - Scavia '25; Colliot-Thélène - Ojanguren '92 and Gabber '90s for perfect k ; conjecture of Grothendieck-Serre '58 for algebraically closed k)

For a field k , a smooth k -gp. scheme G , and a smooth k -variety X , every generically trivial G -torsor E over X trivializes Zariski locally on X , in other words,

$$E(k(x)) \neq \emptyset \Leftrightarrow E(\mathcal{O}_{x,x}) \neq \emptyset \text{ for every } x \in X.$$

Rem. The difficulty is that $E(\mathcal{O}_{x,x}) \not\subseteq E(k(x))$ in general, e.g. for

Ex. If a Serre-Brauer scheme $\begin{smallmatrix} Y \\ \downarrow \\ X \end{smallmatrix}$ is s.t. $Y_{k(x)} \cong \mathbb{P}_{k(x)}^n$, then

7.2/3 $Y \cong \mathbb{P}_X^n$ Zariski locally on X .

Rem. The difficulty is that $E(\mathcal{O}_{x,x}) \not\cong E(k(x))$ in general, e.g.,
for $G = G_m$ and E trivial, $(k(t)_{(t)})^\times \not\cong k(t)^\times$.

$\Rightarrow$ Need to show that E has some $k(x)$ -point defined near a given $x \in X$.

Some simple cases

When G is finite: then E is finite/ X and can use

Lem. For a finite scheme F over a normal, integral scheme S ,
$$F(S) \xrightarrow{\sim} F(k(S))$$

($\Rightarrow E(X) \xrightarrow{\sim} E(k(X))$.)

Pf. Injectivity b/c F/S is separated.

Surjectivity b/c every $k(S)$ -point has the schematic image in F that is a finite, birational modification of S , hence is $\cong S$. $\square$

Rem. Analogously when G_k is constant (= is associated to some abstract gp.).

When G is an abelian variety:

Prop. For an abelian scheme A over a regular integral scheme S ,

$$A(S) \xrightarrow{\sim} A(k(S))$$

($\Rightarrow$ same for every A -torsor by working étale locally on S).

Pf. Injectivity b/c A/S is separated.

Surjectivity b/c $\{ \text{sections of } A \} \leftrightarrow \{ \text{line bundles } \mathcal{L} \text{ on } A^\vee \text{ s.t. } [\mathcal{L}] \in \text{Pic}_{A^\vee}^\circ \cong A \subset \text{Pic}_{A^\vee} \}$

and line bundles on $A^\vee(k(S))$ extend to $A^\vee$ by regularity.

Alternatively: surjectivity b/c every $k(S)$ -point of A has schematic image

$Z \subset A$ s.t. $Z \rightarrow S$ is a projective birational modification. Then

S regular $\Rightarrow$ positive dim'l fibers of Z/S contain $\mathbb{P}^1$'s

Abhyankar's thm

But fibers of A/S contain no $\mathbb{P}^1$'s $\Rightarrow Z/S$ finite $\Rightarrow Z \cong S$. $\square$

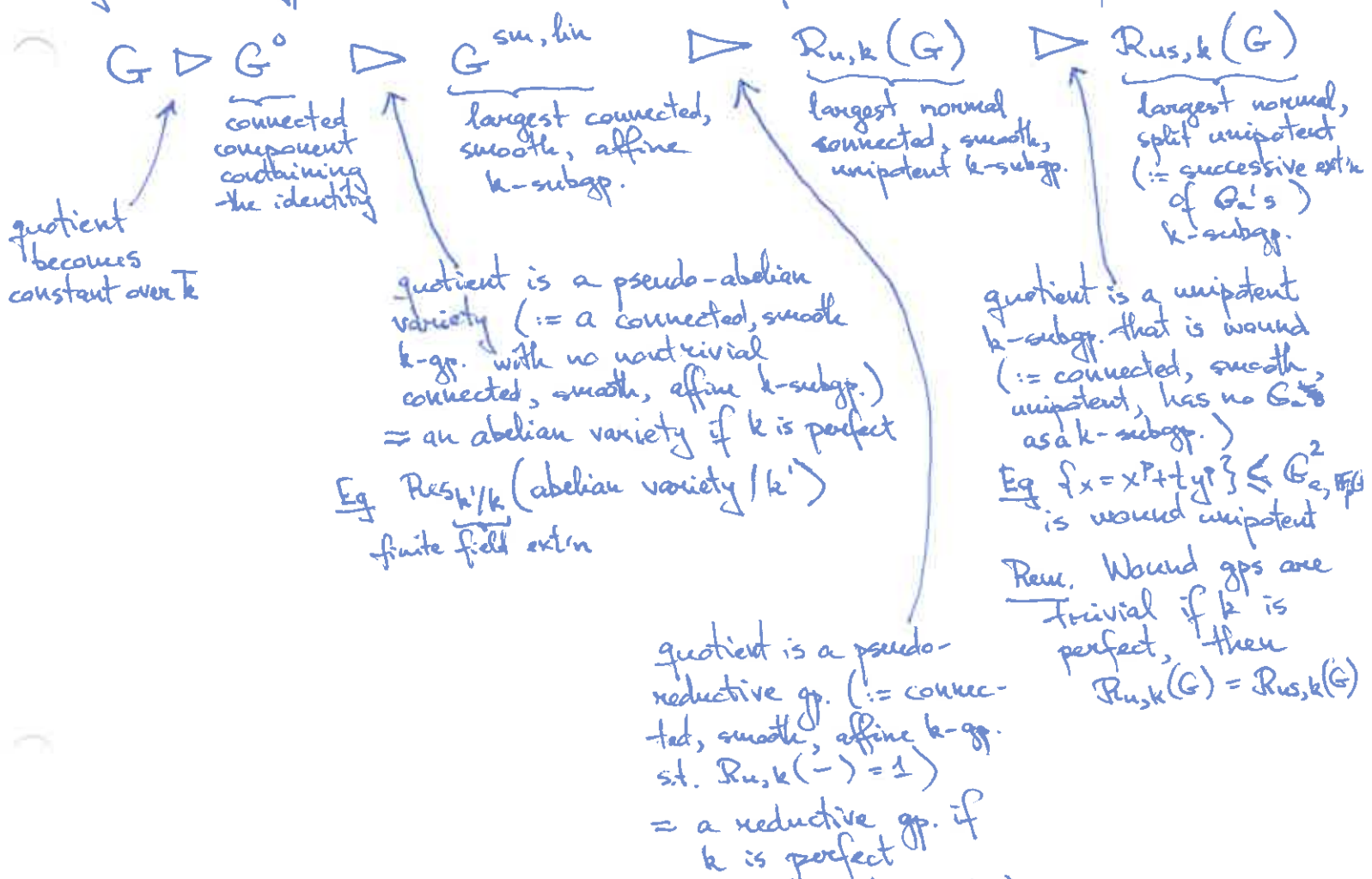
em. (extensions) These simple cases are useful for dévissage: for an extension

$$1 \rightarrow H \rightarrow E \rightarrow Q \rightarrow 1$$

of k -gps, if $Q(X) \xrightarrow{\sim} Q(k(X))$ for every smooth (integral) k -scheme X , then
GS conj. for $H \Rightarrow$ GS conj. for E .

Structure of a general G

Every smooth gp. scheme ~~over~~ G over a field k admits a filtration:



Recall

A reductive k -gp. is a connected, smooth, affine k -gp. H s.t. $R_{u,k}(H_k) = 1$

Eg $GL_n, PGL_n, SO_n, Sp_n, E_6, \dots$

($\Rightarrow$ also $R_{u,k}(H) = 1$, " $\Leftarrow$ " fails for imperfect k)

Simple reductions of the Main Thm.

1. $H^1(\text{affine}, R_{us,k}(G)) = \{*\}$ b/c $H^1(\text{affine}, G_a) = 0$
 $\Rightarrow$ in the Main Thm. wlog $R_{us,k}(G) = 1$.

2. Rem. (extensions) $\Rightarrow$ wlog $G = G^\circ$.

3. For a pseudo-abelian variety A/k , ~~over~~ and a smooth k -sch. X , similarly to Prop., $A(X) \xrightarrow{\sim} A(k(X))$, so
Rem. (extensions) $\Rightarrow$ wlog $G = G^{\text{sm, lin}}$.

Conclusion In the Main Thm., wlog G is quasi-reductive ($\text{:= connected, smooth, affine with } R_{us,k}(G) = 1$) = reductive if k is perfect.