

Recall Goal Thm. For a field k , a smooth k -gp. G , and a smooth (wlog affine, irreducible) k -variety X , every generically trivial G -torsor E over X trivializes over $R := \mathcal{O}_{X,x}$ for each $x \in X$.

Last time: reduced to quasi-reductive G (= to reductive if k is perfect)

Today: use the geometry of X to reformulate the problem.

Presentation lemma

Noether normalization: each affine k -variety Z of dim d admits a finite map $Z \rightarrow \mathbb{A}_k^d$ ($\Rightarrow$ locally X is a smooth curve over $\mathbb{A}_k^d$)

Local structure of smooth maps: each smooth X/k of dim d locally has an étale map $X \rightarrow \mathbb{A}_k^d$ an open $S \subset \mathbb{A}_k^{d-1}$

Gabber-Quillen presentation lemma: let X be an affine, smooth k -variety of dim d and $Z \subsetneq X$ closed of codim > 0 . For each $x \in X$, there are an affine open $x \in X' \subset X$, an affine open $S \subset \mathbb{A}_k^{d-1}$, and a smooth map $\pi: X' \rightarrow S$ of relative dim 1 s.t. $Z \cap X'$ is finite over S .

Idea Compactify $X \subset \bar{X}$, choose

- a hyperplane $H_0 \subset \bar{X}$ not containing x ;
- sufficiently general hypersurfaces $H_1, \dots, H_{d-1}$ of degrees $m_1, \dots, m_{d-1}$ s.t. $\overline{Z} \cap H_1 \cap \dots \cap H_{d-1}$ is finite (if k is infinite, $m_i = 1$ are enough)

Consider the resulting

$$\begin{array}{ccc} \bar{X} \setminus H_0 & \subset & \bar{X} \setminus (H_0 \cap \dots \cap H_{d-1}) \\ \downarrow \pi & & \downarrow \text{given by the equations of the } H_i \\ \mathbb{A}_k^{d-1} & \subseteq & \mathbb{P}_k(1, m_1, \dots, m_{d-1}) \\ & \text{"}z_0 \neq 0\text{"} & \text{weighted projective space (= } \mathbb{P}_k^{d-1} \text{ if all } m_i = 1) \end{array}$$

If the H_i are sufficiently gen'l, a restriction of this π will work. Getting smoothness uses the Bertini thm. $\leadsto$ harder if k is finite. $\square$

Base change: $\begin{array}{ccc} C & \xrightarrow{\quad} & X' \\ \downarrow & \searrow \text{"diag"} & \downarrow \\ \text{Spec } R & \longrightarrow & S \end{array}$ to get

- P.2/4
- a smooth affine relative curve C/R with an $i \in C(R)$;
 - a G -torsor E over C with $i^*(E) \cong E|_R$ s.t. $E|_{C|Z}$ is trivial for some R -finite $Z \subset C$ (wlog containing i).
- in the lemma choose $Z \not\supset X$ s.t. $E|_{X|Z}$ is trivial

Upshot: via the presentation lemma, the Goal thm. reduces to studying G -torsors over C/R that are trivial over $C|Z$.

§ Remaining steps for the Goal thm.

1. Reduce further to when $C = A_R^1$ (today).
2. then triviality away from an R -finite $Z \subset A_R^1 \Rightarrow$ there is a G -torsor $\tilde{E}$ over $\mathbb{P}_R^1$ with $\tilde{E}|_{\{t=0\}} \cong E|_R$ and $\tilde{E}|_{\{t=\infty\}}$ trivial.
3. Study G -torsors over $\mathbb{P}_R^1$ to conclude that $E|_R$ is trivial (remaining lectures).

For 1., will use

Leu. (patching) for every Cartesian square

$$\begin{array}{ccc} Z & \hookrightarrow & U \\ \sim \downarrow & & \downarrow f \text{ flat, affine} \\ f(Z) & \xrightarrow{\text{closed immersion}} & U' \end{array} \quad (*)$$

of Noetherian schemes and every (quasi-) affine, flat, finitely presented U' -group H ,

$$\{H\text{-torsors over } U'\} \xrightarrow{\sim} \{H\text{-torsors over } U\} \times \frac{\{H\text{-torsors over } U'|f(Z)\}}{\{H\text{-torsors over } U|Z\}}$$

in particular, an H -torsor over U' amounts to an H -torsor over U , an H -torsor over $U'|f(Z)$, and an isomorphism between their base changes to $U|Z$.

For us: $U \subset C$ some affine open containing i over Z ,
 $U' = A_R^1$.

Want: find such a U for which there is an étale f fitting into $(*)$

key point: build an embedding $Z \hookrightarrow A_R^1$, it will automatically extend to an étale $f: U \rightarrow A_R^1$ as in $(*)$ for some affine open neighborhood U (exercise).

~~first~~ replace Z by its first infinitesimal neighborhood in C to ~~ensure~~ étaleness

§ Building $\mathbb{Z} \hookrightarrow A_R^1$

Problem: if the residue field k of R is finite, $\mathbb{Z}$ could be "too big" to fit inside A_R^1 , e.g., could have $\#\mathbb{Z}(k) > \#A_R^1(k)$.

Def. There is no finite field obstruction to embedding $\mathbb{Z}$ into A_R^1 if $\#\{z \in \mathbb{Z}_k \mid [k_z:k] = d\} \leq \#\{x \in A_k^1 \mid [k_x:k] = d\}$ for all $d \geq 1$.
(this is automatic if k is infinite.)

Lem. ("Pavlov's finite field trick") For any R -finite $\mathbb{Z}$ and $i \in \mathbb{Z}(R)$,

there is a finite étale $\tilde{\mathbb{Z}} = \text{Spec}(\mathcal{O}_{\mathbb{Z}}[T]/(g(T))) \rightarrow \mathbb{Z}$ s.t.

(i) $\tilde{\mathbb{Z}}$ has no finite field obstruction to embedding it into A_R^1 ;

(ii) i lifts to an $\tilde{i} \in \tilde{\mathbb{Z}}(R)$;

(iii) $g(T)$ is monic ($\Rightarrow \tilde{\mathbb{Z}} \rightarrow \mathbb{Z}$ will automatically lift to a finite étale $\tilde{U} \rightarrow U$ for some affine open $U \subset C$ containing $\mathbb{Z}$).

Pf. Not hard: wlog k finite, the idea is to choose $g(T)$ s.t.

• $g(T)|_i = T$ (monic, residually irreducible polynomial of large deg $N-1 \in R[T]$ will give $\tilde{i}$)

• $g(T)|_{\text{finite residue field } k_z \text{ not in } i(R)} = (\text{monic, irreducible polynomial of deg } N) \in k_z[T]$ $\square$

Upshot wlog there is no finite field obstruction to $\mathbb{Z} \hookrightarrow A_R^1$.

then, since $\mathbb{Z}$ is R -finite, Nakayama's lemma $\Rightarrow$ suffices to build $\mathbb{Z}_k \hookrightarrow A_k^1$.

I.e. wlog $R=k$ is a field.

Moreover: deformation thy. $\Rightarrow$ wlog $\mathbb{Z}$ is reduced.

much easier when k is perfect: then k_z/k are separable and can use the Cohen structure thm.

$\leadsto$ remains to show:

Lem. For a smooth curve C over a field k and a closed point $z \in C$, the extension k_z/k is monogenic (generated by a single elt.), i.e., k_z is also the residue field of some closed point of A_k^1 .

Pf. k_z/k monogenic $\iff$ k_z/k has only finitely many subext'n's l/k
field thy.

C smooth at $z \iff \exists$ open $U \subset C$ containing z and an étale $U \rightarrow A_k^1$

$\leadsto k_z$ is finite separable over a monogenic ext'n k'/k .
show that then k_z/k is also monogenic (small argument), use that

p.4/4 there are finitely many possibilities for $l \cap k'$ to reduce to when $l \cap k' = k$). $\square$

Useful generalization (cf. presentation lemma): for an infinite field k , a smooth, affine, irreducible k -scheme X of $\dim d > 0$, a closed k -subscheme $Z \subsetneq X$, and $x_1, \dots, x_n \in X$, there are an affine open $X' \subset X$ containing neighborhood of the x_i and an étale map $X' \rightarrow \mathbb{A}_k^d$ s.t. $Z \cap X' \rightarrow \mathbb{A}_k^d$ is an ~~closed~~ immersion.