

Generically trivial torsors under algebraic groups

4/9/25, Iasi, Lecture 3
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Last time: reduced the Goal Thm. to showing that for a G -torsor E over $\mathbb{P}_R^1$, if $E|_{\{t=\infty\}}$ is trivial, then so is $E|_{\{t=0\}}$.

Today: the case when R is a field $k \leadsto$ torsors over $\mathbb{P}_k^1$.

Eg. (Grothendieck '57) Every vector bundle over $\mathbb{P}_k^1$ is a direct sum of $\mathcal{O}(i)$'s

Classification Thm. (Bourthier-Ć. - Scavia '25, special cases due to Grothendieck '57, Harder '68, Gille '02, Biswas-Nagaraj '09, Anschütz '18, Wedhorn '24)

For a field k , a smooth k -group scheme G , and a G -torsor E over $\mathbb{P}_k^1$, TFAE:

(i) $s^*(E)$ is trivial for some $s \in \mathbb{P}_k^1(k)$;

(ii) ——— " ——— for every ———;

(iii) $E|_{\mathbb{A}_k^1}$ is trivial;

(iv) E is Zariski locally trivial;

(v) E is generically trivial;

moreover, if the largest connected, smooth, affine k -subgp. $G^{\text{sm}, \text{lin}} \leq G$ is quasi-reductive (e.g., if G is reductive), then

$$H^1(\mathbb{P}_k^1, G) \xleftarrow[\cong]{\text{O}(1)} H^1(BG_{m,k}, G), \quad (*)$$

(i) - (v) are also equivalent to

(vi) $\exists$ a homomorphism $\lambda: G_{m,k} \rightarrow G$ s.t. $E \cong \lambda_*(\mathcal{O}(1))$;

and

$$\begin{aligned} H_{\text{Zar}}^1(\mathbb{P}_k^1, G) &\cong \text{Ker}(H^1(BG_{m,k}, G) \rightarrow H^1(k, G)) \cong \text{Hom}_{k\text{-gp.}}(G_{m,k}, G) / G(k)_{\text{conj}} \\ &\cong \text{Hom}_{k\text{-gp.}}(G_{m,k}, S) / (N_G(S) / Z_G(S))(k) \\ &\quad \uparrow \\ &\quad \text{maximal split torus of } G \end{aligned}$$

Eg. If $G = GL_n$ and $S \cong G_m^n$ is the diagonal torus, then $N_G(S) / Z_G(S) \cong$ the symmetric group, every torsor is Zariski locally trivial, and

$$H^1(\mathbb{P}_k^1, GL_n) \cong \{(i_1, \dots, i_n) \in \mathbb{Z}^n \text{ with } i_1 \leq \dots \leq i_n\}$$

P.2/4 For the Thm.:

1. Reduce to quasi-reductive G as in Lecture 1.
2. Reduce everything to $(*)$. ← acts by $a \cdot (x, y) = (ax, ay)$
3. View $\mathbb{P}_k^1 \cong [(A_k^2 \setminus \{0\}) / G_{m,k}]$ and show $(*)$ in two steps:

$$H^1(\mathbb{P}_k^1, G) \cong H^1([A_k^2 / G_{m,k}], G) \cong H^1(BG_{m,k}, G)$$

extension thm. for torsors under quasi-reductive groups

invariance of $H^1(-, G)$ under stacky Henselian pairs, e.g., under $BG_{m,k} \subset [A_k^2 / G_{m,k}]$

§ Extension Theorem

Extension thm. for reductive groups (Colliot-Thélène - Sansuc '79) For a 2-dimensional regular local ring $(R, \mathfrak{m})$ and a reductive R -group scheme G ($:=$ a smooth, affine R -gp. with connected reductive fibers),

$$\{G\text{-torsors over } \text{Spec}(R)\} \xrightarrow{\sim} \{G\text{-torsors over } \text{Spec}(R) \setminus \{\mathfrak{m}\}\}.$$

Pf. Full faithfulness b/c G is affine and $R \xrightarrow{\sim} \Gamma(\text{Spec}(R) \setminus \{\mathfrak{m}\}, \mathcal{O})$.

Essential surj. for GL_n by the Auslander - Buchsbaum formula:

$$\dim(\text{Supp}(M)) = \text{depth}_{\mathfrak{m}}(M) + \text{pd}_R(M) \quad \text{for finite } R\text{-modules } M.$$

Apply this to $M := \Gamma(\text{Spec}(R) \setminus \{\mathfrak{m}\}, \mathcal{V})$ for a vector bundle $\mathcal{V}$ on $\text{Spec}(R) \setminus \{\mathfrak{m}\}$

$$\text{then } \dim(\text{Supp}(M)) = 2, \text{ depth}_{\mathfrak{m}}(M) \geq 2 \Rightarrow \text{pd}_R(M) = 0$$

↑ b/c M is reflexive ($M^{**} \cong M$)

$\leadsto M$ is a vector bundle / R extending $\mathcal{V}$.

Essential surjectivity for general reductive G : embed $G \hookrightarrow GL_n$, then

$$\{G\text{-torsors over an } R\text{-scheme } S\} \leftrightarrow \{GL_n\text{-torsors } E \text{ over } S \text{ equipped with an } \alpha \in (E/G)(S)\}$$

then E extends by the GL_n case and α extends by the argument for full faithfulness b/c E/G is affine by

Matsushima's Thm. (Alper '14) For a scheme S , a reductive S -gp.

scheme $\tilde{G}$ (e.g., $GL_{n,S}$), and a smooth closed S -subgp. $G \leq \tilde{G}$ with connected fibers, G is reductive if and only if $\tilde{G}/G$ is affine over S . $\square$

Extension thm. for quasi-reductive groups (Bourthier - Č. - Scavia '25)

For a field k , a quasi-reductive k -gp. G , and a 2-dimensional local ring $(R, \mathfrak{m})$ of a smooth k -scheme with $k_{\mathfrak{m}}/k$ separable,

$$\{G\text{-torsors over } \operatorname{Spec}(R)\} \xrightarrow{\sim} \{G\text{-torsors over } \operatorname{Spec}(R) \setminus \{\infty\}\}.$$

Pf. idea Can no longer directly reduce to GL_n b/c GL_n/G is not affine
 G is not reductive. Instead, first prove purity thm's for torsors

- (a) under wound unipotent gps;
- (b) under commutative pseudo-finite groups;

finite type k -gps H with $H(k^s)$ finite
 then use (a) to reduce from quasi-reductive to pseudo-reductive,
 then use (b) to reduce from pseudo-reductive to reductive. $\square$

Lifting under Henselian pairs

Def. A pair (A, I) of a ring A and an ideal $I \subset A$ is

(a) complete if $A \xrightarrow{\sim} \varprojlim_n (A/I^n)$;

(b) Henselian if for every étale, affine A -scheme X , we have $X(A) \twoheadrightarrow X(A/I)$

Rem. Complete $\Rightarrow$ Henselian: if (A, I) is complete, then for every smooth, affine A -scheme X , we have

$$X(A) \cong \varprojlim_n X(A/I^n) \xrightarrow{\twoheadrightarrow} X(A/I)$$

$\uparrow$ "continuity", obvious b/c X is affine $\uparrow$ surjective by deformation thm.

Lifting under Henselian pairs holds beyond étale, affine X :

Thm. (Bouthier-C. '22; special cases due to Grothendieck '72, ...) For a Henselian pair (A, I) and an X that is either

- a smooth A -scheme; or
- a smooth algebraic A -stack with quasi-affine diagonal;

we have $X(A) \twoheadrightarrow X(A/I)$.

Pf. idea Limit arguments + Popescu thm. to reduce to complete (A, I) .
 Then deformation thm. + algebraization results for algebraic stacks due to Hall-Rydh '19. $\square$

Cor. For a Henselian pair (A, I) and a smooth, quasi-affine A -gp. H

$$H^1(A, H) \xrightarrow{\sim} H^1(A/I, H).$$

Pf. Surjectivity: apply the thm. to $X = BH$.

Injectivity: apply the thm. to $X = \operatorname{Isom}_G(E, E')$ for G -torsors E, E' . $\square$

Generalization to stacky Henselian pairs:

thm. (Alper-Hall-Rydh '25) Let $\mathcal{A}$ be an algebraic stack with an affine diagonal and stabilizers of multiplicative type, let $\mathcal{A}/\mathcal{I} \subset \mathcal{A}$ be a closed substack that on coarse moduli spaces is a closed immersion $\mathrm{Spec}(A/\mathcal{I}) \subset \mathrm{Spec}(A)$ for a Henselian pair $(A, \mathcal{I})$. For every smooth morphism ~~$X \rightarrow \mathcal{A}$~~ of algebraic stacks $X \rightarrow \mathcal{A}$ with a quasi-affine diagonal,

$$X(\mathcal{A}) \twoheadrightarrow X(\mathcal{A}/\mathcal{I}).$$

In particular, for every smooth, quasi-affine $\mathcal{A}$ -gp. H , have

$$H^1(\mathcal{A}, H) \xrightarrow{\sim} H^1(\mathcal{A}/\mathcal{I}, H).$$

Ex $\mathcal{A} = [A_k^2 / G_{m,k}]$ with $\mathcal{A}/\mathcal{I} = \mathbb{B}G_{m,k} \subset [A_k^2 / G_{m,k}]$

Then $A = k[x, y]^{G_{m,k}} = k$ with $\mathcal{I} = ((x, y) \overset{\text{"origin"}}{k[x, y]})^{G_{m,k}} = 0$.

$\leadsto (A, \mathcal{I})$ is a Henselian pair.

Get that for every smooth, quasi-affine k -gp. (or even $\mathcal{A}$ -gp.) H ,

$$H^1([A_k^2 / G_{m,k}], H) \xrightarrow{\sim} H^1(\mathbb{B}G_{m,k}, H).$$