

Generically trivial torsors under algebraic groups

5/9/25, Iasi, Lecture 4
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Fix a field k and a smooth k -gp. G .

Thm. (Bouttier-Č. - Scavia '25) For a (semi)local k -alg. $(R, \mathfrak{m})$ and a G -torsor E over $\mathbb{P}_R^1$, if $E|_{\{t=\infty\}}$ is trivial, then so is $E|_{\{t=0\}}$.

Recall (1) Thm. $\Rightarrow$ Zariski local triviality of generically trivial G -torsors
 $\uparrow$ Lecture 2 (Grothendieck-Serre conj. for G)

(2) Lecture 3: settled the case $R=k$, in which E even reduces to the G_m -torsor $\mathcal{O}(1)$ along some k -homomorphism $\lambda: G_{m,k} \rightarrow G$.

Eq (R general, G a torus) For a torus T over a scheme S ,

$$H^1(\mathbb{P}_S^1, T) \cong H^1(S, T) \oplus \underbrace{H^0(S, X_*(T))}_{(\lambda: G_{m,S} \rightarrow T)}$$

inflation of $\mathcal{O}(1)$ to a T -torsor along λ

$\Rightarrow$ every T -torsor $\xrightarrow{E}$ over $\mathbb{P}_S^1$ ~~that~~ with $E|_{\{t=\infty\}}$ trivial has $E|_{\mathbb{A}_S^1}$ trivial

$\Rightarrow$ the Thm. holds for $G=T$.

Initial steps for the Thm.:

(i) Reduce to affine, even quasi-reductive G (key case), similarly to Lecture 1

(ii) ~~Try to~~ Reduce to when R is a field via the geometry of Bun_G :

Prop. For a flat, finitely presented, affine group H over a scheme S , the S -stack

$$\text{Bun}_H: S\text{-scheme } S' \mapsto \{H\text{-torsors on } \mathbb{P}_{S'}^1\}$$

is algebraic, smooth, and the map

$$\text{BH} \longrightarrow \text{Bun}_H$$

H -torsor on an S -scheme $S' \mapsto$ its pullback to an H -torsor on $\mathbb{P}_{S'}^1$

is an open immersion. In particular, for a local ring $(A, \mathfrak{m})$, and

an H -torsor E on $\mathbb{P}_A^1$ descends to R iff $E|_{\mathbb{P}_{k_m}^1}$ descends to k_m .

$\square$ Algebraicity is a generality about restrictions of scalars (Hall-Rydh '19)

Smoothness and open immersion are argued via deformation thry.

(important: $H^1(\mathbb{P}_{S'}^1, \mathcal{O}) = 0$)

$\square$

P.2/3 Upslot: to get the Thm. from the Prop., want a G -torsor E' over $\mathbb{P}_R^1$ with $E'|_{\mathbb{A}_R^1} \simeq E|_{\mathbb{A}_R^1}$ and $E'|_{\{t=\infty\}}$, $E'|_{\mathbb{P}_{km}^1}$ trivial.

atching $\Rightarrow G(R((s)))/G(R[[s]]) \xrightarrow{\text{with } s=t^{-1}} \{E' \text{ with } E'|_{\mathbb{A}_R^1} \simeq E|_{\mathbb{A}_R^1} \text{ and } E'|_{\{t=\infty\}} \text{ trivial}\}$
and likewise with $R=km$.

the field case (Lecture 3) $\Rightarrow E|_{\mathbb{A}_{km}^1}$ is trivial.

$\Rightarrow$ Enough to show: $G(R((s)))/G(R[[s]]) \xrightarrow{?} G(km((s)))/G(km[[s]])$
~~by lifting (last time)~~
Elementary matrices

Def. For a connected, smooth, affine k -gp. ~~G~~ let H , let $H(k)^+ \triangleleft H(k)$ be the (normal) subgp. generated by the $U(k)$ for split unipotent k -subgps $U \leq H$.

the Whitehead group of H is

$$W(k, H) := H(k)/H(k)^+$$

Eg $GL_n(k)^+ \triangleleft GL_n(k)^*$ is the ^{normal} subgp. generated by the elementary matrices $\begin{pmatrix} 1 & & \\ & \ddots & \\ * & & 1 \end{pmatrix}$ with a single nonzero off-diagonal entry and

$$\varinjlim_n W(k, GL_n) \cong K_1(k)$$

$\leadsto$ the Whitehead gp. of H is the "unstable H -valued K_1 -group $K_1^H(k)$ "

Basic properties:

(1) $H(k)^+$ and $W(k, H)$ are functorial in k and H .

(2) $W(k, H) \xrightarrow{\sim} W(k, H/R_{us, k}(H))$

If k is infinite, then largest quasi-reductive quotient of H

(3) for every maximal k -split torus $S \leq H$,

$$H(k) = Z_H(S)(k) H(k)^+, \text{ i.e., } Z_H(S)(k) \twoheadrightarrow W(k, H).$$

L.m. $R((s)) \twoheadrightarrow km((s))$ so also $U(R((s))) \twoheadrightarrow U(km((s)))$ for every split unipotent k -gp. U
"more complicated in reality, but that's the idea"
 $\Rightarrow$ every elt. of $GL_n(km((s)))^+$ lifts to an elt. of $GL_n(R((s)))$.
 $\Rightarrow$ would be done if $W(km((s)))$ was generated by $GL_n(km[[s]])$.

